Independent paths and K_5-subdivisions

Jie Ma*
Xingxing Yu†
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA

Abstract

A well known theorem of Kuratowski states that a graph is planar iff it contains no subdivision of K_5 or $K_{3,3}$. Seymour conjectured in 1977 that every 5-connected nonplanar graph contains a subdivision of K_5. In this paper, we prove several results about independent paths (no vertex of a path is internal to another), which are then used to prove Seymour's conjecture for two classes of graphs. These results will be used in a subsequent paper to prove Seymour's conjecture for graphs containing K_4^-, which is a step in a program to approach Seymour's conjecture.

AMS Subject Classification: 05C38, 05C40, 05C75
Keywords: Subdivision of graph, independent paths, nonseparating path, planar graph

*jiema@math.gatech.edu
†yu@math.gatech.edu; Partially supported by NSA, and by NSFC Project 10628102

1 Introduction

Only finite simple graphs are considered. We follow Diestel [5] for notation and terminology not explicitly defined. In particular, for a graph K we use TK to denote a subdivision of K. Thus, the well known Kuratowski's theorem can be stated as follows: A graph is planar iff it contains no TK_5 or $TK_{3,3}$. It is known that any 3-connected nonplanar graph other than K_5 contains a $TK_{3,3}$. Seymour [16] conjectured in 1977 that every 5-connected nonplanar graph contains a TK_5, which was also posed by Kelmans [10] in 1979.

For convenience, the vertices with degree 4 in a TK_5 are called *branch* vertices. Suppose G is a 5-connected graph and an edge xy of G is contained in three triangles, say xyv_1x, xyv_2x and xyv_3x. Then $G-\{x,y\}$ is 3-connected, and hence contains a cycle C such that $\{v_1,v_2,v_3\} \subseteq C$. Clearly, C and these three triangles form a TK_5 in G with branch vertices x,y,v_1,v_2,v_3.

A graph has an edge in two triangles iff it contains K_4^-, the graph obtained from K_4 by deleting an edge. As a first step in a program to approach Seymour's conjecture, we wish to exclude K_4^-, i.e., to prove it for graphs containing a K_4^-. Note that K_4^--free graphs have nice structural properties; for example, it is shown in [7] that if G is 5-connected and K_4^--free then G contains a contractible edge (see [8] for more results).

It turns out to be quite difficult to find a TK_5 in a 5-connected nonplanar graph containing K_4^-. We shall see in a subsequent paper that given a K_4^- in a 5-connected nonplanar graph, we may be forced to find a TK_5 in which no vertex of this K_4^- is a branch vertex.

The paths $P_1,\ldots,P_k$ are said to be *independent* if for any $1 \le i \ne j \le k$ no vertex of P_i is an internal vertex of P_j. In this paper we prove several results on independent paths, which will be used to prove Seymour's conjecture for two classes of graphs. All these results will be used in a subsequent paper to prove Seymour's conjecture for graphs containing K_4^-.

We use $\emptyset$ to denote both the empty set and the empty graph. Let G be a graph; then $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. By $H \subseteq G$, we mean that H is a subgraph of G. For $X \subseteq V(G)$ or $X \subseteq E(G)$, $G[X]$ denotes the subgraph of G induced by X. For $X \subseteq V(G) \cup E(G)$ or $X \subseteq G$, $G-X$ denotes the graph obtained from G by deleting the vertices in X and those edges in G incident with vertices in X. If $x \in V(G) \cup E(G)$, we write $G-x$ instead of $G-\{x\}$.

We can now state our first result.

Theorem 1.1 *Let G be a 5-connected nonplanar graph and let x_1,x_2,y_1,y_2 be distinct vertices of G such that $G[\{x_1,x_2,y_1,y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Suppose there is an induced path X in $G-x_1x_2$ from x_1 to x_2 such that $G-V(X)$ is 2-connected and $\{y_1,y_2\} \cap V(X) = \emptyset$. Then G contains a TK_5 in which x_1,x_2,y_1,y_2 are branch vertices.*

For subgraphs G and H of a graph, $G \cup H$ and $G \cap H$ denote the *union* and *intersection* of G and H, respectively. We say that G and H are *disjoint* if $V(G) \cap V(H) = \emptyset$. We use $G-H$ instead of $G-V(G\cap H)$. A *separation* of a graph G is a pair (G_1,G_2) of subgraphs of G such that $G = G_1 \cup G_2$, $E(G_1 \cap G_2) = \emptyset$, and $E(G_i) \cup V(G_i - G_{3-i}) \ne \emptyset$ for $i \in \{1,2\}$. If $|V(G_1 \cap G_2)| = k$, then (G_1,G_2) is a *k-separation.*

The following result says that whenever a 5-connected nonplanar graph has a 5-separation and one side of the 5-separation is planar and nontrivial then it contains a TK_5. By an *edge crossing* we mean an intersection of two edges in a drawing of a graph in the plane (vertices are represented by points and edges by polygonal arcs). A drawing of a graph in the plane without edge crossings is also said to be a *planar representation* of that graph.

Theorem 1.2 *Let G be a 5-connected nonplanar graph and let (G_1, G_2) be a 5-separation in G. Suppose $|G_2| \geq 7$ and G_2 has a planar representation in which the vertices of $V(G_1 \cap G_2)$ are incident with a common face. Then G contains a TK_5.*

Another step in our program is to prove that if G is a 5-connected nonplanar graph with a 5-separation (G_1, G_2) such that $|G_i| \geq 2$ for $i = 1, 2$ then G admits a TK_5. This was also suggested by Kawarabayashi.

One of the key ideas in our proof is to find, in a 5-connected graph, an induced path with given ends whose removal results in a graph that is at least 2-connected. This is related to the conjecture of Lovász [13] that there is a minimum integer $c(k) > 0$ such that for any integer $k \geq 1$ and any two vertices u and v in a $c(k)$-connected graph G, there is a path P from u to v in G such that $G - V(P)$ is k-connected. A result of Tutte [20] implies $c(1) = 3$. That $c(2) = 5$ follows from results of Chen, Gould and Yu [3] and Kriesell [12], which are further extended in [4,9].

Let x_1, x_2, y_1, y_2 be vertices of a K_4^- in a 5-connected nonplanar graph G such that $y_1y_2 \notin E(G)$. We show in Section 2 that there is an induced path P in $G - \{x_1x_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ between x_1 and x_2 such that $\{y_1, y_2\} \not\subseteq V(P)$ and $G - V(P)$ is 2-connected. We then prove Theorem 1.1 in Section 3 (the case when $\{y_1, y_2\} \cap V(P) = \emptyset$), using a result of Watkins and Mesner [21] on cycles through three given vertices. (The remaining case when $|\{y_1, y_2\} \cap V(P)| = 1$ is more difficult, and will be proved in another paper with the help of Theorem 1.2.) In Section 4, we prove Theorem 1.2.

We mention several results and problems related to Seymour's conjecture. Mader [14] proved that if G is a simple graph with $n \neq 3$ vertices and at least $3n - 5$ edges then G contains a TK_5, establishing a conjecture of Dirac [6]. Kézdy and McGuiness [11] showed that Seymour's conjecture if true would imply Mader's result. Seymour's conjecture is also related to a conjecture of Hajós (see [1]) that every graph containing no TK_{k+1} is k-colorable. Hajós' conjecture is false for $k \geq 6$ [1] and true for $k = 1, 2, 3$, and remains open for the case $k = 4$ and $k = 5$.

We conclude this section with additional notation and terminology. Let G be a graph. If there is no confusion, we may write $S \subseteq G$ instead of $S \subseteq V(G)$ or $S \subseteq E(G)$, and write $x \in G$ instead of $x \in V(G)$ or $x \in E(G)$. Let $Y \subseteq G$; then $N_G(Y)$, or $N(Y)$ if G is understood, denotes the set of vertices in $V(G) - V(Y)$ adjacent to vertices in $V(Y)$. If $Y = \{y\} \subseteq V(G)$, then we use $N_G(y)$ or $N(y)$ instead of $N_G(\{y\})$ or $N(\{y\})$. Let T be a set of 2-element subsets of $V(G)$; then $G + T$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup T$. If $T = \{\{x, y\}\}$, we write $G + xy$ instead of $G + \{\{x, y\}\}$.

Given a path P in a graph and $x, y \in V(P)$, xPy denotes the subpath of P between x and y (inclusive). The *ends* of the path P are the vertices of the minimum degree in P, and the other vertices of P are its *internal* vertices. A path P with ends u and v is also said to be *from u to v* or *between u and v*. Let H_1 and H_2 be subgraphs of G; a path P in G is an H_1-H_2 path if P has one end in H_1 and another in H_2, and is otherwise disjoint from $H_1 \cup H_2$. A path P from x to y in a graph G is said to be *internally disjoint* from $H \subseteq G$ if $P \cap H \subseteq \{x, y\}$.

Let G be a graph. A set $S \subseteq V(G)$ is a *k-cut* or a *cut* of size k in G, where k is a positive integer, if $|S| = k$ and G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = S$ and

$V(G_i - S) \neq \emptyset$ for $i \in \{1, 2\}$. If $v \in V(G)$ and $\{v\}$ is a cut of G, then v is said to be a *cut vertex* of G.

For a subgraph H of a graph G, an *H-bridge* of G is a subgraph of G, say B, for which there exists a component D of $G - V(H)$ such that B is induced by the edges which are either contained in D or from D to H. The vertices in H that are neighbors of D are called the *attachments* of this H-bridge. For $S \subseteq V(G)$, the $G[S]$-bridges of G are also called S-bridges.

2 Nonseparating paths

In this section we prove three lemmas, two on nonseparating paths and one on independent paths. A *nonsepararting* path in a graph G is a path P such that $G - V(P)$ is connected. We need the following concept of connectivity.

Definition 2.1 *Let G be a graph and $S \subseteq V(G)$, and let k be a positive integer. We say that G is (k, S)-connected if, for any cut T of G with $|T| < k$, every component of $G - T$ contains a vertex from S.*

We also need a result of Seymour [17]; equivalent formulations can be found in [2, 18, 19].

Theorem 2.2 (*Seymour*) *Let G be a graph and let s_1, s_2, t_1, t_2 be distinct vertices of G. Then either G contains disjoint paths from s_1 to s_2 and from t_1 to t_2, or there exist pairwise disjoint sets $A_i \subseteq V(G)$ ($k \geq 0$ and $1 \leq i \leq k$), such that*

(*a*) *for $i \neq j$, $N(A_i) \cap A_j = \emptyset$,*

(*b*) *for $1 \leq i \leq k$, $|N(A_i)| \leq 3$, and*

(*c*) *the graph, obtained from G by (for each i) deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, can be drawn in a closed disc with no edge crossings such that s_1, t_1, s_2, t_2 occur on the boundary of the disc in cyclic order.*

As a consequence, if G is $(4, \{s_1, s_2, t_1, t_2\})$-connected, then either G has disjoint paths from s_1 to s_2 and from t_1 to t_2, or G can be drawn in a closed disc in the plane with no edge crossings such that s_1, t_1, s_2, t_2 occur on the boundary in cyclic order.

Let G be a graph; a *chain of blocks* in G is a sequence $B_1 B_2 ... B_k$ such that each B_i is a block of G, $B_i \cap B_j = \emptyset$ when $|i - j| \geq 2$, and $|V(B_i \cap B_{i+1})| = 1$ for $1 \leq i \leq k - 1$. If $k = 1$ and $x, y \in V(B_1)$, or if $k \geq 2$ and $x \in V(B_1 - B_2)$ and $y \in V(B_k - B_{k-1})$, then $B_1 B_2 ... B_k$ is said to be a chain of blocks *from x to y* (or *from x*, or *from y*).

The lemma below allows one to modify an existing path to a good nonseparating path.

Lemma 2.3 *Let G be a graph and let x_1, x_2, y_1, y_2 be distinct vertices of G such that G is $(5, \{x_1, x_2, y_1, y_2\})$-connected. Suppose X is an induced path in G from x_1 to x_2, and H is a chain of blocks in $G - V(X)$ from y_1 to y_2. Then precisely one of the following holds:*

(*i*) *$H = y_1 y_2$ and $G - y_1 y_2$ can be drawn in a closed disc in the plane without edge crossings such that x_1, y_1, x_2, y_2 occur on the boundary of the disc in this cyclic order.*

(*ii*) *There is an induced path X' from x_1 to x_2 such that $H \subseteq G - V(X')$, and $G - V(X')$ is a chain of blocks from y_1 to y_2.*

Proof. First, we may assume that if $y_1y_2 \in E(G)$ then $H \neq y_1y_2$; in particular, $|V(H)| \geq 3$. For, suppose $y_1y_2 \in E(G)$ and $H = y_1y_2$. If $G - y_1y_2$ contains disjoint paths X', Y from x_1, y_1 to x_2, y_2, respectively, then we see that in $G - X'$, $\{y_1, y_2\}$ is contained in a block H' which contains the cycle $H \cup Y$; so we may replace X, H by X', H', respectively. On the other hand, (i) follows from Lemma 2.2 and the assumption that G is $(5, \{x_1, x_2, y_1, y_2\})$-connected.

We now choose such X and H that

(1) H is maximal (under subgraph containment), and

(2) subject to (1), the number of components of $G - V(X)$ is minimum.

Next, we show that $G - V(X)$ is connected. For, suppose there is a component of $G - V(X)$ disjoint from H, and let D be such a component. Let v_1, v_2 denote the neighbors of D on X with v_1Xv_2 maximal. (D has at least 5 neighbors on X; so $v_1v_2 \notin E(G)$.) Since G is $(5, \{x_1, x_2, y_1, y_2\})$-connected, $v_1Xv_2 - \{v_1, v_2\}$ contains a neighbor of some component of $G - V(X)$ other than D, say C. Now let X' be obtained from X by deleting $v_1Xv_2 - \{v_1, v_2\}$ and adding an induced path in $G[V(D) \cup \{v_1, v_2\}]$ from v_1 to v_2. Let D' denote the union of those components of $D - X'$ with no neighbor in $v_1Xv_2 - \{v_1, v_2\}$. (Possible $D' = \emptyset$.) We choose X' so that

(3) D' is minimal.

If $D' = \emptyset$ then $(D - X') \cup C \cup (v_1Xv_2 - \{v_1, v_2\})$ is contained in a component of $G - X'$, and the number of components of $G - V(X')$ is smaller than $G - V(X)$, contradicting to (2) (since H will not get smaller). So we may assume $D' \neq \emptyset$. Let $D_1, \ldots, D_k$ be the components of D'. Let a_i, b_i $(1 \leq i \leq k)$ denote the neighbors of D_i in $v_1X'v_2$ with $a_iX'b_i$ maximal. Since G is $(5, \{x_1, x_2, y_1, y_2\})$-connected, $\{a_i, b_i, v_1, v_2\}$ in not a cut in G, so there exists $c_i \in a_iX'b_i - \{a_i, b_i\}$ such that c_i has a neighbor in $D - (X' \cup D_i)$ or in $v_1Xv_2 - \{v_1, v_2\}$. If c_i has a neighbor that belongs to $v_1Xv_2 - \{v_1, v_2\}$, or that is not in D' but is contained in a component of $D - X'$, then let X'' be obtained from X' by deleting $a_iX'b_i - \{a_i, b_i\}$ and adding an induced path between a_i and b_i in $G[V(D_i) \cup \{a_i, b_i\}]$; it is easy to see that X'' contradicts the choice of X' in (3). Thus, for any $1 \leq i \leq k$, $N(a_iX'b_i - \{a_i, b_i\}) \subseteq X' \cup D'$. Therefore, $\bigcup_{i=1}^{k} a_iX'b_i$ is a subpath of $v_1X'v_2$; let a, b denote its ends. Now $\{a, b, v_1, v_2\}$ is not a cut in G, so there exists $c \in aX'b - \{a, b\}$ such that c has a neighbor in $v_1Xv_2 - \{v_1, v_2\}$, or in a component of $D - X'$ that is not a component of D'. Then there exists some $1 \leq i \leq k$ such that $c \in a_iX'b_i - \{a_i, b_i\}$, which is a contradiction since we have shown that $N(a_iX'b_i - \{a_i, b_i\}) \subseteq X' \cup D'$.

Having shown that $G - V(X)$ is connected, we may now assume that $G - V(X) \neq H$; as otherwise $X' := X$ is the desired path for (ii). Let D be an arbitrary H-bridge of $G - V(X)$ with $V(D) \cap V(H) = \{v\}$. Let v_1, v_2 denote the neighbors of $D - v$ on X with v_1Xv_2 maximal.

Suppose there are independent paths Q, R in G from $v_1Xv_2 - \{v_1, v_2\}$ to distinct vertices of H which are also internally disjoint from $D \cup X \cup H$. Then let X' be obtained from X by deleting $v_1Xv_2 - \{v_1, v_2\}$ and adding an induced path in $G[V(D - v) \cup \{v_1, v_2\}]$ from v_1 to v_2. Clearly, in $G - V(X')$ the chain of blocks from y_1 to y_2 contains $H \cup Q \cup R$, contradicting (1).

So all paths from $v_1Xv_2 - \{v_1, v_2\}$ to H internally disjoint from $D \cup X \cup H$ must end at the same vertex, say u, in H. Moreover, at least one such path has length at least 2; for otherwise, because $|V(H)| \geq 3$, $\{v, u, v_1, v_2\}$ would be a 4-cut in G (contradicting the assumption that G is $(5, \{x_1, x_2, y_1, y_2\})$-connected). Hence there exists some H-bridge C of $G - V(X)$ such that $V(C \cap H) = \{u\}$ and $C - u$ contains a neighbor of $v_1Xv_2 - \{v_1, v_2\}$. Let u_1, u_2 denote the neighbors of $C - u$ on X with u_1Xu_2 maximal.

Suppose $v_1Xv_2 \subseteq u_1Xu_2$. Then since G is $(5, \{x_1, x_2, y_1, y_2\})$-connected, $\{u, v, u_1, u_2\}$ is not a cut in G. Hence, since $|V(H)| \geq 3$, there is a path R in G from $u_1Xu_2 - \{u_1, u_2\}$ to $H - \{u, v\}$ internally disjoint from $C \cup D \cup X \cup H$. Let X' be obtained from X by deleting $u_1Xu_2 - \{u_1, u_2\}$ and adding an induced path in $G[V(C - u) \cup \{u_1, u_2\}]$ from u_1 to u_2. Clearly, in $G - V(X')$ the chain of blocks from y_1 to y_2 contains $H \cup R$ and part of $D \cup u_1Xu_2$, contradicting (1).

If $u_1Xu_2 \subseteq v_1Xv_2$, then the same argument above (by simply exchanging the roles of C, u, u_1, u_2 with D, v, v_1, v_2, respectively) gives a contradiction to (1).

So neither v_1Xv_2 nor u_1Xu_2 is contained in the other. By symmetry we may assume that $x_1, u_1, v_1, u_2, v_2, x_2$ occur on X in this order. Since G is $(5, \{x_1, x_2, y_1, y_2\})$-connected, $\{u, v, u_1, v_2\}$ is not a cut in G. Hence, since $|V(H)| \geq 3$, there is a path R in G from $r \in V(u_1Xv_2) - \{u_1, v_2\}$ to $H - \{u, v\}$ internally disjoint from $C \cup D \cup X \cup H$. Note that $r \notin v_1Xv_2 - \{v_1, v_2\}$, and so $r \in u_1Xu_2 - \{u_1, u_2\}$. Let X' be obtained from X by deleting $u_1Xu_2 - \{u_1, u_2\}$ and adding an induced path in $G[V(C - u) \cup \{u_1, u_2\}]$ from u_1 to u_2. In $G - V(X')$, the chain of block from y_1 to y_2 contains $H \cup R$ and part of $D \cup u_1Xu_2$, contradicting (1). ∎

We now prove that in a 5-connected nonplanar graph containing K_4^-, one can find a good nonseparating path.

Lemma 2.4 *Let G be a 5-connected nonplanar graph and x_1, x_2, y_1, y_2 be distinct vertices of G such that $G[\{x_1, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(G)$. Then there is an induced path X in $G - \{x_1x_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ from x_1 to x_2 such that $G - V(X)$ is 2-connected and $\{y_1, y_2\} \not\subseteq V(X)$.*

Proof. For convenience, let $G' := G - \{x_1x_2, x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$. Since G is 5-connected, G' is $(5, \{x_1, x_2, y_1, y_2\})$-connected, y_1 has a neighbor y different from x_1, x_2, y_2, and $G' - \{y_1, y\}$ contains an induced path X from x_1 to x_2. Let B denote the block of $G' - V(X)$ containing y_1y. It is possible that $B = y_1y$.

Because G is nonplanar, (i) of Lemma 2.3 cannot occur. So viewing B as a chain of blocks from y_1 to y and applying Lemma 2.3, we conclude that there is an induced path X' in G' from x_1 to x_2 such that $B \subseteq G' - V(X')$, and $G' - V(X')$ is a chain of blocks from y_1 to y.

Suppose $V(G') - V(X') \neq \{y_1, y\}$. Then, since $y_1y \in B \subseteq G' - V(X')$ and $G - V(X')$ is a chain of blocks from y_1 to y, $G' - V(X')$ is 2-connected. Clearly, $\{y_1, y_2\} \not\subseteq G' - V(X')$. Note that $G' - V(X') = G - V(X')$. So X' is a desired path.

Therefore, we may assume $V(G') - V(X') = \{y_1, y\}$. In this case, since G is 5-connected and $y_1y_2 \notin E(G)$, there is a vertex $x \in V(X') - \{x_1, x_2, y_2\}$. Hence, because X' is induced, x has at most four neighbors: y_1, y and two vertices on X'. This contradicts the assumption that G is 5-connected. ∎

From Lemma 2.4 we see that in order to prove Seymour's conjecture for graphs with K_4^-, it suffices to prove Theorem 1.1 (when $\{y_1, y_2\} \cap V(X) = \emptyset$) and deal with the case when $|\{y_1, y_2\} \cap V(X)| = 1$. (The later will be done in another paper.) Before we prove Theorem 1.1, we need a lemma about independent paths.

Lemma 2.5 *Let G be a graph and $S \subseteq V(G)$ such that G is $(4, S)$-connected. Assume that there exist $a_1, a_2 \in S$, $a \in V(G) - S$, and two independent paths in $G - (S - \{a_1, a_2\})$ from a to a_1, a_2 respectively. Then there exist four independent paths in G from a to distinct vertices in S, one from a to a_1 and another from a to a_2.*

Proof. Since G is $(4, S)$-connected, $|S| \geq 4$; and it follows from Menger's theorem that there exist four independent paths P_i, $i = 1, 2, 3, 4$, in G from a to $b_i \in S$, respectively, and internally disjoint from S. For convenience, let $P := \bigcup_{i=1}^4 P_i$. We choose P_1, P_2, P_3, P_4 so that $\ell := |\{a_1, a_2\} \cap \{b_1, b_2, b_3, b_4\}|$ is maximum.

Note that $0 \leq \ell \leq 2$. If $\ell = 2$ then P_1, P_2, P_3, P_4 are the desired paths. So we may assume $\ell = 0$ or $\ell = 1$. By assumption, let Q_i $(i = 1, 2)$ be independent paths in $G - (S - \{a_1, a_2\})$ from a to a_i, and let $x_i \in V(Q_i \cap P)$ such that $V(a_i Q_i x_i \cap P) = \{x_i\}$.

Suppose $\ell = 0$. Without loss of generality, we may assume that $x_2 \in P_1$. Then the paths $aP_1x_2 \cup x_2Q_2a_2, P_2, P_3, P_4$ contradict the choice of P_1, P_2, P_3, P_4 (the maximality of ℓ).

So $\ell = 1$, and we may assume, without loss of generality, that $a_1 = b_1$ and $a_2 \notin \{b_1, b_2, b_3, b_4\}$.

We may assume $x_2 \in P_1$; otherwise, assume without loss of generality that $x_2 \in P_2$, and then $P_1, aP_2x_2 \cup x_2Q_2a_2, P_3, P_4$ are the desired paths for the lemma. We may also assume $x_1 \in P_1$; for, otherwise, assume (without loss of generality) that $x_1 \in P_2$, and then $aP_2x_1 \cup x_1Q_1a_1, aP_1x_2 \cup x_2Q_2a_2, P_3, P_4$ are the desired paths for the lemma.

Now suppose there exists $i \in \{1, 2\}$ such that $Q_i \cap (P_2 \cup P_3 \cup P_4) = \{a\}$. We only deal with $i = 1$; the case when $i = 2$ is symmetric. Suppose then that $Q_1 \cap (P_2 \cup P_3 \cup P_4) = \{a\}$. Then we may assume $Q_2 \cap (P_2 \cup P_3 \cup P_4) \neq \{a\}$, since otherwise, Q_1, Q_2, P_2, P_3 are the desired paths for the lemma. So let $y_2 \in V(Q_2) \cap V(P_2 \cup P_3 \cup P_4)$ such that $y_2 \neq a$ and $V(a_2Q_2y_2) \cap V(P_2 \cup P_3 \cup P_4) = \{y_2, a\}$, and we may assume without loss of generality that $y_2 \in P_2$. Now $Q_1, aP_2y_2 \cup y_2Q_2a_2, P_3, P_4$ are the desired paths for the lemma.

Thus, we may assume that $Q_i \cap (P_2 \cup P_3 \cup P_4) \neq \{a\}$ for $i \in \{1, 2\}$. Let $y_i \in V(Q_i) \cap V(P_2 \cup P_3 \cup P_4)$ such that $y_i \neq a$ and $V(a_iQ_iy_i) \cap V(P_2 \cup P_3 \cup P_4) = \{y_i\}$. Note that $a_iQ_ix_i \subseteq a_iQ_iy_i$ and $x_i \neq y_i$. Let $x_i' \in V(x_iQ_iy_i \cap P_1)$ such that $x_i'P_1a$ is minimum. Note that $x_1' \neq x_2'$. Suppose $x_1' \in a_1P_1x_2'$. Without loss of generality assume $y_1 \in P_2$. Then $aP_2y_1 \cup y_1Q_1a_1, aP_1x_2' \cup x_2'Q_2a_2, P_3, P_4$ are the desired paths. Now assume $x_2' \in a_1P_1x_1'$, and $y_2 \in P_2$ (without loss of generality). Then $aP_1x_1' \cup x_1'Q_1a_1, aP_2y_2 \cup y_2Q_2a_2, P_3, P_4$ are the desired paths. ∎

3 Proof of Theorem 1.1

We need a result of Watkins and Mesner [21] that characterizes those graphs in which no cycle contains a set of three specified vertices. This result is also used in [23] in the reduction of Hajós' conjecture to 4-connected graphs. See Figure 1 for an illustration.

Theorem 3.1 (*Watkins and Mesner*) *Let R be a 2-connected graph and let y_1, y_2, v be three distinct vertices of R. Then there is no cycle through y_1, y_2 and v in R if, and only if, one of the following statements holds.*

(*i*) *There exists a 2-cut S in R and, for $u \in \{y_1, y_2, v\}$, there exist pairwise disjoint subgraphs D_u of $R - S$ such that $u \in D_u$ and each D_u is a union of components of $R - S_u$.*

(*ii*) *For $u \in \{y_1, y_2, v\}$, there exist 2-cuts S_u of R and pairwise disjoint subgraphs D_u of R, such that $u \in D_u$, each D_u is a union of components of $R - S_u$, $S_{y_1} \cap S_{y_2} \cap S_v = \{z\}$, and $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_v - \{z\}$ are pairwise disjoint.*

(*iii*) *For $u \in \{y_1, y_2, v\}$, there exist pairwise disjoint 2-cuts S_u in R and pairwise disjoint subgraphs D_u of $R - S_u$ such that $u \in D_u$, D_u is a union of components of $R - S_u$, and $R - V(D_{y_1} \cup D_{y_2} \cup D_v)$ has precisely two components, each containing exactly one vertex from S_u.*

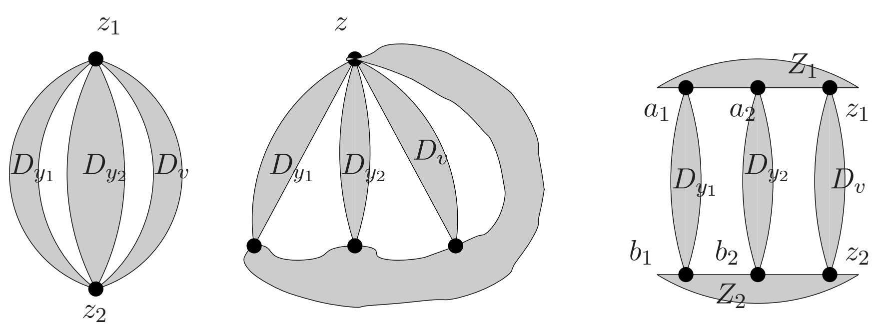

Figure 1: The subgraphs D_u in R, with $u \in \{y_1, y_2, v\}$.

Proof of Theorem 1.1. If there exists $x \in V(X - \{x_1, x_2\})$ such that $\{x, y_1, y_2\}$ is contained in some cycle, say D, in $G - V(X - x)$, then $D \cup X \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G, with branch vertices x_1, x_2, y_1, y_2, x. Hence we may assume that

(1) for any $x \in V(X - \{x_1, x_2\})$, no cycle in $G - V(X - x)$ contains $\{x, y_1, y_2\}$.

Let v denote the neighbor of x_2 in X. Since $|N(v)| \geq 5$ and X is an induced path in G, $|N(v) - V(X)| \geq 3$. Let $R = G - V(X - v)$. Clearly, R is 2-connected. By (1), $\{y_1, y_2, v\}$ is not contained in any cycle in R. Hence, (i) or (ii) or (iii) of Theorem 3.1 holds (see Figure 1). We choose X so that

(2) $D_{y_1} \cup D_{y_2} \cup D_v$ is maximal.

We shall treat all three cases, (i), (ii) and (iii), simultaneously. For this we need some notation. If (i) occurs let $S_v := S = \{z_1, z_2\}$, and if (ii) or (iii) occurs let $S_v = \{z_1, z_2\}$. Let Z_i denote the component of $R - V(D_{y_1} \cup D_{y_2} \cup D_v)$ containing z_i. If (i) occurs then let $a_1 = a_2 = z_1$ and $b_1 = b_2 = z_2$; and if (iii) occurs let $S_{y_1} = \{a_1, b_1\}$ and $S_{y_2} = \{a_2, b_2\}$ such that $a_1, a_2 \in Z_1$ and $b_1, b_2 \in Z_2$. If (ii) occurs let $S_{y_1} = \{a_1, b_1\}$ and $S_{y_2} = \{a_2, b_2\}$, and we know either $z = z_1 = a_1 = a_2$ or $z = z_2 = b_1 = b_2$ (we do not fix this notation for the purpose of symmetry in arguments to follow).

Note that if (i) occurs then $Z_i := \{z_i\}$ for $i = 1, 2$; and if (ii) or (iii) occurs then by (2) and the fact that R is 2-connected, $S_u \cap V(Z_i)$, for $u \in \{y_1, y_2, v\}$, are not cuts in Z_i. Also note that if (ii) occurs, then $Z_1 = Z_2$, or $Z_1 \neq Z_2$ and $Z_1 = \{z_1\}$, or $Z_1 \neq Z_2$ and $Z_2 = \{z_2\}$. So the case when (ii) occurs with $Z_1 \neq Z_2$ may also be viewed as that when (iii) occurs. We now prove the following claim.

(3) For any $x \in V(Z_1)$, $Z_1 - z_2$ has independent paths A_1, A_2 from $\{x, z_1\}$ to a_1, a_2, and $Z_1 - z_2$ has a path A between a_1 and a_2 and independent from z_1; for any $x \in V(Z_2)$, $Z_2 - z_1$ has independent paths B_1, B_2 from $\{x, z_2\}$ to b_1, b_2, and $Z_2 - z_1$ has a path B between b_1 and b_2 and independent from z_2.

Since the two statements of (3) are symmetric, we only prove the existence of B_1, B_2, B. If $b_1 = b_2 = z_2$ then we simply take $B_1 = B_2 = B = \{z_2\}$. So we may assume b_1, b_2, z_2 are pairwise distinct (and hence (ii) or (iii) occurs).

If B_1, B_2 do not exist, then $Z_2 - z_1$ has a cut vetex z_2' separating $\{b_1, b_2\}$ from $\{x, z_2\}$; and we see that $S_{y_1}, S_{y_2}, S_v' := \{z_1, z_2'\}$ contradict (2).

Now suppose the path B does not exist. Then z_2 is a cut vertex in $Z_2 - z_1$ separating b_1 and b_2. If (ii) occurs then $S = \{z_1, z_2\}$ is a cut in R such that y_1, y_2, v are contained in different components of $G - S$, contradicting (2). If (iii) occurs then $S_{y_1}' := \{a_1, z_2\}$, $S_{y_2}' := \{a_2, z_2\}$ and S_v are cuts in R contradicting (2). This completes the proof of (3).

Since R is 2-connected, for each $u \in \{y_1, y_2, v\}$, $R[D_u \cup S_u]$ is a chain of blocks between the vertices of S_u, and there is a path P_u in $R[D_u \cup S_u]$ between the vertices of S_u and containing u. Let P_u^i denote the subpath of P_u from u to $S_u \cap Z_i$.

(4) We may assume that $N(z_i) \cap (X - \{x_1, x_2, v\}) = \emptyset$ for $i = 1, 2$, and that D_v is connected.

Suppose (4) fails. By symmetry, we may assume $N(z_1) \cap (X - \{x_1, x_2, v\}) \neq \emptyset$. Then we can find a path P from z_1 to $a \in V(x_1Xv) - \{x_1, v\}$ and internally disjoint from $X \cup P_{y_1} \cup P_{y_2} \cup P_v$, as follows. If $N(z_1) \cap V(X - \{x_1, x_2, v\}) \neq \emptyset$ let $a \in N(z_1) \cap V(x_1Xv - \{x_1, v\})$ and let $P := z_1a$. If D_v is not connected, then let D be a component of D_v such that $v \notin D$. Since G is 5-connected, there exists $a \in N(D) \cap V(x_1Xv - \{x_1, v\})$. Let P be a path in $R[V(D) \cup S_v \cup \{a\}] - z_2$ from z_1 to a.

Choose A_1, A_2, B as in (3) with $x = z_1$. Then $(A_1 \cup P_{y_1}^1) \cup (A_2 \cup P_{y_2}^1) \cup (P_v^1 \cup vx_2) \cup (P \cup aXx_1) \cup (P_{y_1}^2 \cup B \cup P_{y_2}^2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_1. This proves (4).

Note that $N(D_v) \subseteq S_v \cup X$. Let $u \in N(D_v) \cap V(X)$ with x_1Xu minimal, and let $u' \in N(u) \cap V(D_v)$. Since $\{z_1, z_2, u, x_2\}$ is not a cut in G, there exists an edge cc' with $c \in uXx_2 - \{x_2, u\}$ and $c' \in (D_{y_1} \cup D_{y_2} \cup Z_1 \cup Z_2) - \{z_1, z_2\}$. Note that $c \neq v$, for otherwise S_v is not a 2-cut in R separating D_v from $D_{y_1} \cup D_{y_2} \cup Z_1 \cup Z_2$. So $c \in uXv - \{u, v\}$. Since X is induced, $u' \neq v$. Hence by (4), let $Q_{u'}$ denote a path in D_v from u' to $w \in V(P_v)$ such that $Q_{u'} \cap P_v = \{w\}$. By symmetry, we may assume $w \in P_v^2$. Note $w \neq z_2$, since D_v is connected.

(5) We may assume that $N(c) \cap V(Z_1) = \emptyset$ when $Z_1 \neq Z_2$ or when $b_1 = b_2 = z_2$, and we may assume that if $x \in N(c) \cap V(D_{y_i})$ then for any path P_x in $R[D_{y_i} \cup S_{y_i}]$ from x to P_{y_i}, P_x intersects $P_{y_i}^2 - y_i$ first.

First, suppose $x \in N(c) \cap Z_1$ and $Z_1 \neq Z_2$ or $b_1 = b_2 = z_2$. By (4), $x \neq z_1$ and $x \neq z_2$; and so $Z_1 \neq \{z_1\}$. Let A_1, A_2 be the paths as in (3), and by symmetry we may assume $x \in A_1$ and $a_1 \in A_1$. Let $B = \{z_2\}$ if $Z_1 = Z_2$ and $b_1 = b_2 = z_2$, and otherwise let B be the path as in (3). Then $(vP_v^2 w \cup Q_{u'} \cup u'u \cup uXx_1) \cup vx_2 \cup (vXc \cup cx \cup A_1 \cup P_{y_1}^1) \cup (P_v^1 \cup A_2 \cup P_{y_2}^1) \cup (P_{y_1}^2 \cup B \cup P_{y_2}^2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, v.

Now suppose $x \in N(c) \cap V(D_{y_i})$, and there is a path R_i in $R[D_{y_i} \cup S_{y_i}]$ from x to $x' \in V(P_{y_i}^1)$ such that $R_i \cap P_{y_i} = \{x'\}$. Without loss of generality, assume $i = 1$. Choose B as in (3). Also by (3), let A_2 be a path in $Z_1 - z_2$ from z_1 to a_2 and independent from $A_1 = \{a_1\}$. Note that if $Z_1 \neq Z_2$ then $B \cap (A_1 \cup A_2) = \emptyset$; and if $Z_1 = Z_2$ then either $a_1 = a_2 = z_1$ (with $A_1 = A_2 = \{z_1\}$) or $b_1 = b_2 = z_2$ ($B = \{z_2\}$),and we have $B \cap (A_1 \cup A_2) = \emptyset$ as well.

Then $(vP_v^2 w \cup Q_{u'} \cup u'u \cup uXx_1) \cup vx_2 \cup (vXc \cup cx \cup R_1 \cup x'P_{y_1}^1 y_1) \cup (P_v^1 \cup A_2 \cup P_{y_2}^1) \cup (P_{y_1}^2 \cup B \cup P_{y_2}^2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, v.

(6) We may assume that $N(c) \cap V(D_{y_1} \cup D_{y_2} \cup Z_1 \cup Z_2) = \{c'\}$.

Otherwise, we may assume by (4) and (5) that there exists $a \in N(c) \cap V(D_{y_1} \cup D_{y_2} \cup (Z_2 - \{z_1, z_2\}))$ such that $a \neq c'$.

Suppose $\{a, c'\} \subseteq Z_2 - \{z_1, z_2\}$. Then only (ii) or (iii) can occur. First, assume $Z_2 - z_1$ has disjoint paths B_1', B_2' from $\{a, c'\}$ to b_1, b_2, respectively. Then $b_1 \neq b_2$, and hence, either $Z_1 \neq Z_2$ or $Z_1 = Z_2$ and $a_1 = a_2 = z_1$. So let $A := \{z_1\}$ if $Z_1 = Z_2$ and $a_1 = a_2 = z_1$; otherwise let A be the path in (3). Now $\{y_1, y_2, c\}$ is contained in the cycle $B_1' \cup B_2' \cup P_{y_1} \cup P_{y_2} \cup A \cup \{c, cc', ca\}$ in $G - V(X - c)$, contradicting (1). Therefore, we may assume that such paths B_1', B_2' do not exist for any choice of $\{a, c'\}$ with $\{a, c'\} \subseteq Z_2 - \{z_1, z_2\}$. Then by (2), there is a cut vertex z in $Z_2 - z_1$ separating $N(c) \cap Z_2$ from $\{b_1, b_2\}$. Suppose $Z_1 \neq Z_2$. Since R is 2-connected, z must separate $\{b_1, b_2\}$ from $(N(z) \cap Z_2) \cup \{z_2\}$. But then $S_v' := \{z, z_1\}, S_{y_1}, S_{y_2}$ contradict (2). So $Z_1 = Z_2$, and hence by (5), $a_1 = a_2 = z_1$. If z_2 is in the z-bridge of Z_2 that also contains $N(c) \cap Z_2$, then $S_v' := \{z, z_1\}, S_{y_1}, S_{y_2}$ contradict (2). So in $Z_2 - z_1$, z separates $\{b_1, b_2, z_2\}$ from $N(c) \cap Z_2$. Note that $c' \neq z$ or $a \neq z$. Without loss of generality, assume $c' \neq z$. Then since R is 2-connected, Z_2 contains disjoint paths R_1, R_2 from z_1, b_1 to c', b_2, respectively. Now $(R_1 \cup cc' \cup x_1Xc) \cup (P_v^1 \cup vx_2) \cup P_{y_1}^1 \cup P_{y_2}^1 \cup (P_{y_1}^2 \cup R_2 \cup P_{y_2}^2) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z_1.

So we may assume $\{a, c'\} \not\subseteq Z_2 - \{z_1, z_2\}$. Then $N(c) \cap V(D_{y_1} \cup D_{y_2}) \neq \emptyset$ (by (4) when $Z_1 = Z_2$, and by (4) and (5) when $Z_1 \neq Z_2$). So we may assume by symmetry that $c' \in D_{y_1}$. Let $P_{c'}$ be a path in D_{y_1} from c' to $c'' \in V(P_{y_1})$ such that $P_{c'} \cap P_{y_1} = \{c''\}$. By (5), $c'' \in P_{y_1}^2 - y_1$.

Suppose $a \in D_{y_2} \cup (Z_2 - \{z_1, z_2\})$. If $a \in D_{y_2}$ then by (5) there exists a path P_a in D_{y_2} from a to $a' \in V(P_{y_2})$ such that $P_a \cap P_{y_2} = \{a'\}$ and $a' \in P_{y_2}^2 - y_2$. Recall the path A from (3). Now $\{c, cc', ca\} \cup P_{c'} \cup c''P_{y_1}a_1 \cup A \cup a_2P_{y_2}a' \cup P_a$ is a cycle in $G - V(X - c)$ containing $\{y_1, y_2, c\}$, contradicting (1). If $a \in Z_2 - \{z_1, z_2\}$, then there is a path P_a in $Z_2 - z_1$ from a to b_2. Again, $\{c, cc', ca\} \cup P_{c'} \cup c''P_{y_1}a_1 \cup A \cup P_{y_2} \cup P_a$ is a cycle in $G - V(X - c)$ containing $\{y_1, y_2, c\}$, contradicting (1).

So we may assume $a \in D_{y_1}$. Since $R[S_{y_1} \cup D_{y_1}]$ is a chain of blocks, it has disjoint paths $P_a, P_{c'}$ from a, c' to $a', c'' \in V(P_{y_1})$ such that $P_a \cap P_{y_1} = \{a'\}$ and $P_{c'} \cap P_{y_1} = \{c''\}$. By (5), we have $\{a', c''\} \subseteq P_{y_1}^2$. Without loss of generality, we may assume that $a' \in b_1P_{y_1}^2c''$. If $Z_1 = Z_2$ and $z_1 = a_1 = a_2$ let $A = \{z_1\}$ and B be as in (3); if $Z_1 = Z_2$ and $b_1 = b_2 = z_2$ then let $B = \{z_2\}$ and A be as in (3); and if $Z_1 \neq Z_2$ let A and B be as in (3). Then

$\{c, cc', ca\} \cup P_{c'} \cup c''P_{y_1}a_1 \cup A \cup P_{y_2} \cup B \cup a'P_{y_1}b_1 \cup P_a$ is a cycle in $G - V(X - c)$ containing $\{y_1, y_2, c\}$, contradicting (1) and completing the proof of (6).

By (6), $N(c) \cap V(D_v - S_v) \neq \emptyset$. Without loss of generality and by (5) and (6), we may assume that $c' \in D_{y_1} \cup (Z_2 - \{z_1, z_2\})$. Moreover, if $c' \in D_{y_1}$, let $P_{c'}$ be a path in D_{y_1} from c' to $c'' \in V(P_{y_1})$ such that $P_{c'} \cap P_{y_1} = \{c''\}$ and $c'' \in P^2_{y_1} - y_1$ (by (5)).

(7) We may assume that v is a cut-vertex of $R[S_v \cup D_v] - z_1z_2$ separating z_2 from $(N(c) \cap V(D_v)) \cup \{z_1\}$.

Otherwise, since $R[S_v \cup D_v] - z_1z_2$ is a chain of blocks from z_1 to z_2, there is a path P_a in $R[S_v \cup V(D_v)] - \{v, z_1\}$ from some $a \in N(c) \cap V(D_v - v)$ to z_2.

Suppose $c' \in D_{y_1}$. If $Z_1 = Z_2$ and $a_1 = a_2 = z_1$ let $A = \{z_1\}$ and P be a path in $Z_2 - z_1$ from z_2 to b_2; if $Z_1 = Z_2$ and $b_1 = b_2 = z_2$ let $P = \{z_2\}$ and A be as in (3); and if $Z_1 \neq Z_2$, let A be as in (3) and P be a path in $Z_2 - z_1$ from z_2 to b_2. It is easy to see that $\{c, cc', ca\} \cup P_a \cup P \cup P_{y_2} \cup A \cup a_1P_{y_1}c'' \cup P_{c'}$ is a cycle in $G - V(X - c)$ containing $\{y_1, y_2, c\}$, contradicting (1).

So $c' \in Z_2 - \{z_1, z_2\}$. Then by (5), $Z_1 \neq Z_2$, or $Z_1 = Z_2$ and $a_1 = a_2 = z_1$. If $Z_1 \neq Z_2$ let A, B_1, B_2 be as in (3); and otherwise let $A = \{z_1\}$, and let B_1, B_2 be as in (3) with $c' \in B_1$ and $b_1 \in B_1$. Now $\{c, cc', ca\} \cup P_a \cup B_2 \cup P_{y_2} \cup A \cup P_{y_1} \cup B_1$ is a cycle in $G - V(X - c)$ containing $\{y_1, y_2, c\}$, contradicting (1).

Let T denote the v-bridge of $R[S_v \cup D_v] - z_1z_2$ containing z_2. Recall u, u', and $u' \neq v$ (before (5)). Since $w \in P^2_v \subseteq T$, $u' \in T - v$. Since G is 5-connected and by the choice of u, $G[V(T) \cup V(uXx_2)]$ is $(5, V(uXx_2) \cup \{z_2, v\})$-connected, and so $G' := G[V(T) \cup V(uXx_2 - u)]$ is $(4, V(uXx_2 - u) \cup \{z_2, v\})$-connected. So by Lemma 2.5, there exist four independent paths P_1, P_2, P_3, P_4 in G' from u' to $(uXx_2 - u) \cup \{z_2, v\}$ such that P_1 ends at z_2, P_2 ends at v, and P_3, P_4 both end in $uXx_2 - \{u, v\}$. Since $vx_2 \in E(X)$, we may assume that P_3 ends at $x' \in V(uXv) - \{u, v\}$.

Suppose $c' \in D_{y_1}$. If $Z_1 = Z_2$ and $a_1 = a_2 = z_1$ let $A = \{z_1\}$ and let B'_2 be a path in $Z_2 - z_1$ from z_2 to b_2; if $Z_1 = Z_2$ and $b_1 = b_2 = z_2$ let $B'_2 = \{z_2\}$ and A be as in (3); and if $Z_1 \neq Z_2$ let A be as in (3) and B'_2 be a path in Z_2 from z_2 to b_2. Now $(u'u \cup uXx_1) \cup (P_1 \cup B'_2 \cup P^2_{y_2}) \cup (P_2 \cup vx_2) \cup (P_3 \cup x'Xc \cup P_{c'} \cup c''P^2_{y_1}y_1) \cup (P^1_{y_1} \cup A \cup P^1_{y_2}) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, u'.

So we may assume $c' \in Z_2 - \{z_1, z_2\}$. Then by (5), $Z_1 \neq Z_2$, or $Z_1 = Z_2$ and $a_1 = a_2 = z_1$. If $Z_1 = Z_2$ and $a_1 = a_2 = z_1$ let $A = \{z_1\}$; if $Z_1 \neq Z_2$ let A be defined as in (3). Let B_1, B_2 be defined as in (3) (with c' as x). If $z_2 \in B_2$ and $c' \in B_1$, then $(u'u \cup uXx_1) \cup (P_1 \cup B_2 \cup P^2_{y_2}) \cup (P_2 \cup vx_2) \cup (P_3 \cup x'Xc \cup cc' \cup B_1 \cup b_1P^2_{y_1}y_1) \cup (P^1_{y_1} \cup A \cup P^1_{y_2}) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, u'. If $z_2 \in B_1$ and $c' \in B_2$, then $(u'u \cup uXx_1) \cup (P_1 \cup B_1 \cup P^2_{y_1}) \cup (P_2 \cup vx_2) \cup (P_3 \cup x'Xc \cup cc' \cup B_2 \cup b_2P^2_{y_2}y_2) \cup (P^1_{y_1} \cup A \cup P^1_{y_2}) \cup G[\{x_1, x_2, y_1, y_2\}]$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, u'. ∎

4 Planar graphs

In this section we prove Theorem 1.2, using an approach similar to that in [22] where rooted K_4-subdivisions are considered. This result will be useful in situations where we force a 5-separation in a 5-connected nonplanar graph such that one side of the separation is planar.

It is well known that every face of a 2-connected plane graph is bounded by a cycle. The *outer* cycle of a 2-connected plane graph is the boundary of its infinite face. In a plane graph, two vertices are said to be *cofacial* if they are incident with a common face. Let C be a cycle in a plane graph and $x, y \in V(C)$; if $x \neq y$ we use xCy to denote the path in C clockwise from x to y, and if $x = y$ then xCy represents the path consisting of the vertex $x = y$. For a vertex x in a graph, we use $d(x)$ to denote the degree of x.

Lemma 4.1 *Let G be a graph drawn in a closed disc in the plane without edge crossings, and let a_1, a_2, a_3, a_4, a_5 be distinct vertices of G on the boundary of the disc, and let $A := \{a_1, a_2, a_3, a_4, a_5\}$. Suppose G is $(5, A)$-connected and $|V(G)| \geq 7$. Then $G - A$ is 2-connected, and $G - A$ is not spanned by its outer cycle. Moreover, for each $w \in V(G) - A$ which is not on the outer cycle of $G - A$, all vertices of G that are cofacial with w induce a cycle in $G - A$.*

Proof. Without loss of generality we may assume that a_1, a_2, a_3, a_4, a_5 lie on the boundary of the disc in the clockwise order listed. Since G is $(5, A)$-connected, $d(v) \geq 5$ for all $v \in G - A$.

First, we claim that $G - A$ is connected and has no cut vertex. Otherwise, there is a separation (G_1, G_2) in $G - A$ of order at most 1 such that $G_1 - G_2 \neq \emptyset$ and $G_2 - G_1 \neq \emptyset$. Note that $|N(G_1 - G_2) \cap A| \geq 4$ since otherwise $V(G_1 \cap G_2) \cup (N(G_1 - G_2) \cap A)$ is a cut in G separating A from G_1, contradicting the assumption that G is $(5, A)$-connected. Therefore, by planarity, we may assume (with appropriate notation change) that a_1, a_2, a_3, a_4 all have neighbors in $G_1 - G_2$. Then by planarity we see that $\{a_4, a_5, a_1\} \cup V(G_1 \cap G_2)$ is a cut in G separating G_2 from A, a contradiction (since G is $(5, A)$-connected).

Therefore, $G - A \cong K_2$ or $G - A$ is 2-connected. Indeed, $G - A$ must be 2-connected. For, suppose $G - A \cong K_2$. Let $V(G - A) = \{a, b\}$. Then $|N(a) \cap A| \geq 4$, or else $(N(a) \cap A) \cup \{b\}$ is a cut of size at most 4 separating A from b, a contradiction. Similarly, $|N(b) \cap A| \geq 4$. However, this contradicts planarity.

Let C denote the outer cycle of $G - A$. We now show that $V(G - A) \neq V(C)$. For, suppose $V(G - A) = V(C)$; we will derive a contradiction. If $|V(C)| = 3$, then each vertex in $V(C)$ has at least 3 neighbors in A, which is not possible due to planarity. So $|V(C)| \geq 4$. Since all edges of $G - A$ are on C or inside C, it follows from planarity that there are two vertices on C with degree 2 in $G - A$, say u and v, such that $uv \notin E(G)$. Since G is $(5, A)$-connected and by planarity, we may assume $a_1, a_2, a_3 \in N(u)$ and $a_3, a_4, a_5 \in N(v)$; and hence no other vertex of $G - A$ has degree 2, and each edge of $G - A$ not on C joins $uCv - \{u, v\}$ to $vCu - \{u, v\}$. Since G is $(5, A)$-connected and $ua_3, va_3 \in E(G)$, $|N(z) \cap V(C)| \geq 4$ for all $z \in uCv - \{u, v\}$. Let w be the neighbor of u in $uCv - \{u, v\}$, and let w_1, w_2 denote neighbors of w on $vCu - \{v, u\}$ with v, w_1, w_2, u on vCu in order and w_1Cw_2 maximal. Let w_2', w_2'' be the neighbors of w_2 in w_1Cu. Then by planarity and the fact that $d(w_2) \geq 5$, $N(w_2) = \{w_2', w_2'', w, a_1, a_5\}$. Because $d(w_1) \geq 5$ and $a_1 \notin N(w_1)$ (by planarity), there exists $x \in wCv - \{w, v\}$ such that $x \in N(w_1)$. Then we may pick $y \in xCv - v$ such that yCv minimal and y has a neighbor in $vCw_1 - v$. By planarity and the fact $d(y) \geq 5$, $|N(y) \cap V(C)| \geq 4$. Let y_1, y_2 denote neighbors of y on $vCw_1 - v$ with v, y_1, y_2, w_1 on vCw_1 in order. Let y_1', y_1'' be the neighbors of y_1 in vCw_1. Then by planarity, $N(y_1) \subseteq \{y, y_1', y_1'', a_5\}$, contradicting $d(y_1) \geq 5$.

Let $w \in V(G - A)$ such that $w \notin C$. Then, since G is $(5, A)$-connected and by planarity, the vertices of G that are cofacial with w induce a cycle in $G - A$. ∎

Lemma 4.2 *Let G be a connected graph drawn in a closed disc in the plane without edge crossings, let a_1, a_2, a_3, a_4, a_5 be distinct vertices of G on the boundary of the disc, and let $A = \{a_1, a_2, a_3, a_4, a_5\}$. Suppose G is $(5, A)$-connected and $|V(G)| \geq 7$, and assume G has no 5-separation (G_1, G_2) such that $A \subseteq G_1$ and $|V(G)| > |V(G_2)| \geq 7$. Let $w \in V(G) - A$ such that the vertices of G cofacial with w induce a cycle C_w in $G - A$. Then there exist four paths $P_1, \dots, P_4$ from w to A such that*

(i) for $1 \leq i < j \leq 4$, $V(P_i \cap P_j) = \{w\}$, and

(ii) for $1 \leq i \leq 4$, $|V(P_i \cap C_w)| = 1$.

Proof. By assumption, we have

(1) G has no 5-separation (G_1, G_2) such that $A \subseteq G_1$ and $|V(G)| > |V(G_2)| \geq 7$.

By Lemma 4.1, $G - A$ is 2-connected. So $|V(G) - A| \geq 3$. Hence by (1), each a_i has at least two neighbors in $G - A$, and so G is 2-connected. Let C denote the outer cycle of G, and let C' denote the outer cycle of $G - A$. By Lemma 4.1 again, there exists $w \in V(G) - A$ such that the vertices of G which are cofacial with w induce a cycle C_w and $C_w \subseteq G - A$. By planarity, $w \notin C'$.

By Menger's theorem, there exist four paths $Q_1, \dots, Q_4$ from w to A such that $V(Q_i \cap Q_j) = \{w\}$ for $1 \leq i \neq j \leq 4$, and for each i (by planarity, we may assume that) $Q_i \cap C_w$ is a path. Let $\alpha(Q_1, Q_2, Q_3, Q_4)$ denote the number of Q_i such that $|V(Q_i \cap C_w)| \geq 2$. We choose such Q_1, Q_2, Q_3, Q_4 that

(2) $\alpha(Q_1, Q_2, Q_3, Q_4)$ is minimum.

We may assume that the notation is such that $a_i \in Q_i$ for $i = 1, \dots, 4$, and that a_1, a_2, a_3, a_4 occur on the boundary of the disc in clockwise order (a_5 could be anywhere on C). Let $w_i, v_i \in V(Q_i)$ such that $ww_i \in Q_i$ and $V(v_i Q_i a_i \cap C_w) = \{v_i\}$.

If $\alpha(Q_1, Q_2, Q_3, Q_4) = 0$, then $P_i := Q_i$, $1 \leq i \leq 4$, are the desired paths. So we may assume without loss of generality that $|V(Q_1 \cap C_w)| \geq 2$. By symmetry, we may further assume that $v_1 \in w_1 C_w w_2$. See Figure 2 for an illustration. We may also assume that w has no neighbor in $w_1 C_w v_1 - w_1$; for otherwise let w' be a neighbor of w in $w_1 C_w v_1 - w_1$ with $w' C v_1$ minimal, and we may replace Q_1 with $w' Q_1 a_1 + \{w, ww'\}$.

For $1 \leq i \leq 4$, let H_i denote the maximal subgraph of G contained in the closed region in the plane with boundary $Q_i \cup Q_{i+1} \cup a_i C a_{i+1} \cup w_i C_w w_{i+1}$, where $Q_5 = Q_1$, $a_5 = a_1$ and $w_5 = a_1$. Let S_1 denote the vertices of G, each of which is cofacial with some vertex of $w_1 C_w v_1 - w_1$. Then

(3) $S_1 \cap V(v_4 C_w w_1 - w_1) = \emptyset$, and $S_1 \cap V(v_4 Q_4 a_4 - v_4) \neq \emptyset$.

If $S_1 \cap V(v_4 C_w w_1 - w_1) \neq \emptyset$, then there exist $x \in V(v_4 C_w w_1 - w_1)$ and $y \in V(w_1 C_w v_1 - w_1)$ such that $\{x, y, w\}$ is a cut in G separating w_1 from $\{a_1, a_2, a_3, a_4\}$. Since $a_5 \notin C_w$, $w_1 \neq a_5$. So $\{x, y, w, a_5\}$ is a cut in G separating w_1 from A, contradicting the assumption that G is $(5, A\})$-connected. So $S_1 \cap V(v_4 C_w w_1 - w_1) = \emptyset$.

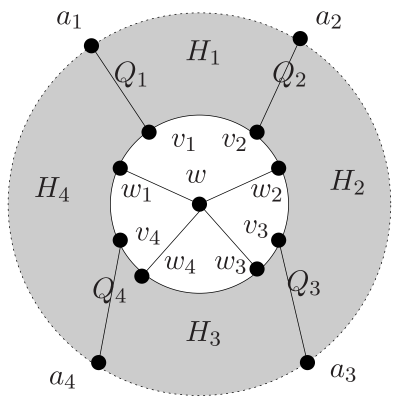

Figure 2: Structure of G around w.

Now suppose $S_1 \cap V(v_4Q_4a_4 - v_4) = \emptyset$. Then by planarity H_4 has a path Q_1' from w_1 to a_1 disjoint from $Q_4 \cup (C_w - w_1)$. Now $\alpha(w_1Q_1'a_5 + \{w, ww_1\}, Q_2, Q_3, Q_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2). So $S_1 \cap V(v_4Q_4a_4 - v_4) \neq \emptyset$, completing the proof of (3).

Let S_4 denote the vertices of G each of which is cofacial with a vertex in $S_1 \cap V(v_4Q_4a_4 - v_4)$. Then

(4) $S_4 \cap V(v_3C_wv_4) = \emptyset$, and $S_4 \cap V(v_3Q_3a_3 - v_3) \neq \emptyset$.

Suppose there exists $u \in S_4 \cap V(v_3C_wv_4)$. Then there exist $u_4 \in S_1 \cap V(v_4Q_4a_4 - v_4)$ and $v \in V(w_1C_wv_1 - w_1)$ such that u and u_4 are cofacial, and u_4 and v are cofacial. Note that $\{u, u_4, v, w\}$ is a cut in G; so, since G is $(5, A)$-connected, $\{u, u_4\} \subseteq C$ and $a_5 \in uCu_4$, or $\{u_4, v\} \subseteq C$ and $a_5 \in u_4Cv$. If $w_1a_5 \notin E(G)$, then the cut $\{u, u_4, v, w, a_5\}$ contradicts (1) (as w_1 has at least 5 neighbors); if $w_1a_5 \in E(G)$ then $\alpha(ww_1a_5, Q_2, Q_3, Q_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2). Hence, $S_4 \cap V(v_3C_wv_4) = \emptyset$.

Now assume $S_4 \cap V(v_3Q_3a_3 - v_3) = \emptyset$. Then by planarity and by the fact that $S_4 \cap V(v_3C_wv_4) = \emptyset$, there is a path Q_4' in $H_3 - (S_1 \cap V(Q_4))$ from v_4 to a_4 disjoint from Q_3 and $C_w - v_4$. Moreover, by (3) and planarity, $H_4 - V(Q_4')$ has a path Q_1' from w_1 to a_1 disjoint from $C_w - w_1$ (which necessarily contains $S_1 \cap V(Q_4)$). Then $\alpha(Q_1' + \{w, ww_1\}, Q_2, Q_3, Q_4' \cup wQ_4v_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2) and completing the proof of (4).

Let S_3 denote the vertices of G each of which is cofacial with a vertex in $S_4 \cap V(v_3Q_3a_3 - v_3)$. Then

(5) $S_3 \cap V(v_2C_wv_3) = \emptyset$, and $S_3 \cap V(v_2Q_2a_2 - v_2) \neq \emptyset$.

First, suppose there exists $u \in S_3 \cap V(v_2C_wv_3)$. Then there exist $u_3 \in S_4 \cap V(v_3Q_3a_3 - v_3)$, $u_4 \in S_1 \cap V(v_4Q_4a_4 - v_4)$, and $v \in V(w_1C_wv_1 - w_1)$ such that u and u_3 are cofacial, u_3 and u_4 are cofacial, and u_4 and v are cofacial. Choose u, u_3, u_4, v so that uC_wv_3, $v_3Q_3u_3$, $v_4Q_4u_4$, and w_1C_wv are minimal (in the order listed).

Let H_2' denote the $\{u, u_3\}$-bridge of H_2 containing uC_wv_3; let H_3' denote the $\{u_3, u_4\}$-bridge of H_3 containing $v_3C_wv_4$; and let H_4' denote the $\{u_4, v\}$-bridge of H_4 containing v_4C_wv. Note that $\{u, u_3, u_4, v, w\}$ is a cut in G; so by (1), $a_5 \in H_i'$ for some $2 \leq i \leq 4$.

Suppose $a_5 \in H'_2$. Then in $H_2 - Q_2$ there is a path Q'_3 from v_3 to a_5 disjoint from $S_4 \cap V(v_3Q_3u_3 - v_3)$ and $(C_w - v_3) \cup u_3Q_3a_3$ (by minimality of uC_wv_3 and $v_3Q_3u_3$). In $H_3 - Q'_3$ there is a path Q'_4 from v_4 to a_3 disjoint from $S_1 \cap V(v_4Q_4u_4 - v_4)$ and $(C_w - v_4) \cup u_4Q_4a_4$ (by (4) and minimality of $v_4Q_4u_4$). In $H_4 - V(Q'_4)$ there is a path Q'_1 from w_1 to a_4 disjoint from $C_w - w_1$ (by (3) and minimality of w_1C_wv). Then $\alpha(Q'_1 + \{w, ww_1\}, Q_2, Q'_3 \cup wQ_3v_3, Q'_4 \cup wQ_4v_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2).

Now assume $a_5 \in H'_3$. In $H_2 - Q_2$ there is a path Q'_3 from v_3 to a_3 disjoint from $S_4 \cap V(v_3Q_3u_3 - \{u_3, v_3\})$ and $C_w - v_3$ (by minimality of uC_wv_3 and $v_3Q_3u_3$). In $H_3 - Q'_3$ there is a path Q'_4 from v_4 to a_5 disjoint from $S_1 \cap V(v_4Q_4u_4 - v_4)$ and $(C_w - v_4) \cup u_4Q_4a_4$ (by (4) and minimality of $v_4Q_4u_4$). In $H_4 - V(Q'_4)$ there is a path Q'_1 from w_1 to a_4 disjoint from $C_w - w_1$ (by (3) and minimality of w_1C_wv). Then $\alpha(Q'_1 + \{w, ww_1\}, Q_2, Q'_3 \cup wQ_3v_3, Q'_4 \cup wQ_4v_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2).

Finally, assume $a_5 \in H'_4$. In $H_2 - Q_2$ there is a path Q'_3 from v_3 to a_3 disjoint from $S_4 \cap V(v_3Q_3u_3 - \{u_3, v_3\})$ and $C_w - v_3$ (by minimality of uC_wv_3 and $v_3Q_3u_3$). In $H_3 - Q'_3$ there is a path Q'_4 from v_4 to a_4 disjoint from $S_1 \cap V(v_4Q_4u_4 - \{v_4, u_4\})$ and $C_w - v_4$ (by (4) and minimality of $v_4Q_4u_4$). In $H_4 - V(Q'_4)$ there is a path Q'_1 from w_1 to a_5 disjoint from $C_w - w_1$ (by (3) and minimality of w_1C_wv). Then $\alpha(Q'_1 + \{w, ww_1\}, Q_2, Q'_3 \cup wQ_3v_3, Q'_4 \cup wQ_4v_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2). This proves $S_3 \cap V(v_2C_wv_3) = \emptyset$.

We now prove $S_3 \cap V(v_2Q_2a_2 - v_2) \neq \emptyset$. For, otherwise, $H_2 - V(Q_2)$ has a path Q'_3 from v_3 to a_3 disjoint from $S_4 \cap V(Q_3)$ and $C_w - v_3$ (since $S_3 \cap V(v_2C_wv_3) = \emptyset$). In $H_3 - Q'_3$ there is a path Q'_4 from v_4 to a_4 disjoint from $S_1 \cap V(Q_4)$ and $C_w - v_4$ (by (4)). In $H_4 - V(Q'_4)$ there is a path Q'_1 from w_1 to a_1 disjoint from $C_w - w_1$ (by (3)). Now $\alpha(Q'_1 + \{w, ww_1\}, Q_2, Q'_3 \cup wQ_3v_3, Q'_4 \cup wQ_4v_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2) and completing the proof of (5).

Let S_2 denote the vertices of G each of which is cofacial with a vertex in $S_3 \cap V(Q_2)$. Then

(6) $S_2 \cap V(v_1C_wv_2) \neq \emptyset$.

Suppose $S_2 \cap V(v_1C_wv_2) = \emptyset$. Then $S_2 \cap V(v_1Q_1a_1 - v_1) \neq \emptyset$. For, otherwise, in $H_1 - Q_1$ there is a path Q'_2 from v_2 to a_2 disjoint from $S_3 \cap V(Q_2)$ and $C_w - v_2$. In $H_2 - Q'_2$ there is a path Q'_3 from v_3 to a_3 disjoint from $S_4 \cap V(Q_3)$ and $C_w - v_3$ (by (5)). In $H_3 - Q'_3$ there is a path Q'_4 from v_4 to a_4 disjoint from $S_1 \cap V(Q_4)$ and $C_w - v_4$ (by (4)). In $H_4 - Q'_4$ there is a path Q'_1 from w_1 to a_1 disjoint from $C_w - w_1$ (by (3)). Now $\alpha(Q'_1 + \{w, ww_1\}, Q'_2 \cup wQ_2v_2, Q'_3 \cup wQ_3v_3, Q'_4 \cup wQ_4v_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2).

Thus, let $u_1 \in S_2 \cap V(v_1Q_1a_1 - v_1)$. Then there exists $u_2 \in S_3 \cap V(v_2Q_2a_2 - v_2)$ such that u_2 and u_1 are cofacial, there exists $u_3 \in S_4 \cap V(v_3Q_3a_3 - v_3)$ such that u_3 and u_2 are cofacial, there exists $u_4 \in S_1 \cap V(v_4Q_4a_4 - v_4)$ such that u_4 and u_3 are cofacial, and there exists $v \in V(w_1C_wv_1 - w_1)$ such that u_4 and v are cofacial. For $i = 1, 2, 3$, define H'_i as the $\{u_i, u_{i+1}\}$-bridge of H_i containing $v_iC_wv_{i+1}$. Define H'_4 as the $\{v, u_4\}$-bridge of H_4 containing $v_4C_wv_1$.

Then H'_1 contains a path Q'_2 from v_2 to u_1 disjoint from $S_3 \cap V(Q_2)$ and $C_w - v_2$ (since we assume $S_2 \cap V(v_1C_wv_2) = \emptyset$). $H'_2 - Q'_2$ contains a path Q'_3 from v_3 to u_2 disjoint from $S_4 \cap V(Q_3)$ and $C_w - v_3$ (by (5)). $H'_3 - Q'_3$ contains a path Q'_4 from v_4 to u_3 disjoint from $S_1 \cap V(Q_4)$ and $C_w - v_4$ (by (4)). $H'_4 - Q'_4$ contains a path Q'_1 from w_1 to u_4 disjoint from $C_w - w_1$ (by (3)). Now $\alpha((Q'_1 + \{w, ww_1\}) \cup u_4Q_4a_4, wQ_4v_4 \cup Q'_4 \cup u_3Q_3a_3, wQ_3v_3 \cup Q'_3 \cup u_2Q_2a_2, wQ_2v_2 \cup Q'_2 \cup u_1Q_1a_1) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2).

By (6) and by the definitions of S_i ($1 \le i \le 4$), we may let $u \in V(v_1C_wv_2)$ and $u_2 \in V(v_2Q_2a_2 - v_2) \cap S_3$ such that u and u_2 are cofacial and, subject to this, uC_wv_2 and $v_2Q_2u_2$ are minimal. Let $u_3 \in V(v_3Q_3a_3 - v_3) \cap S_4$ such that u_2 and u_3 are cofacial and, subject to this, $v_3Q_3u_3$ is minimal. Let $u_4 \in V(v_4Q_4a_4 - v_4) \cap S_1$ such that u_4 and u_3 are cofacial and, subject to this, $v_4Q_4u_4$ is minimal. Let $v \in V(w_1C_wv_1 - w_1)$ such that v and u_4 are cofacial.

Let H'_1 denote the $\{u, u_2\}$-bridge of H_1 containing uC_wv_2; let H'_2 denote the $\{u_2, u_3\}$-bridge of H_2 containing $v_2C_wv_3$; let H'_3 denote the $\{u_3, u_4\}$-bridge of H_3 containing $v_3C_wv_4$; and let H'_4 denote the $\{u_4, v\}$-bridge of H_4 containing v_4C_wv.

(7) $a_5 \notin H'_i$ for $1 \le i \le 4$.

The proof of (7) is similar to that of (5). First, suppose $a_5 \in H'_1$. Then there is a path Q'_2 in H_1 from v_2 to a_5 disjoint from $S_3 \cap V(v_2Q_2u_2 - v_2)$ and $(C_w - v_2) \cup u_2Q_2a_2$ (by minimality of uC_wv_2 and $v_2Q_2u_2$). In $H_2 - Q'_2$ there is a path Q'_3 from v_3 to a_2 disjoint from $S_4 \cap V(v_3Q_3u_3 - v_3)$ and $(C_w - v_3) \cup u_3Q_3a_3$ (by (5) and minimality of $v_3Q_3u_3$). In $H_3 - Q'_3$ there is a path Q'_4 from v_4 to a_3 disjoint from $S_1 \cap V(v_4Q_4u_4 - v_4)$ and $(C_w - v_4) \cup u_4Q_4a_4$ (by (4) and minimality of $v_4Q_4a_4$). In $H_4 - Q'_4$ there is a path Q'_1 from w_1 to a_4 disjoint from $C_w - w_1$ and Q_1 (by (3) and minimality of w_1C_wv). Then $\alpha(Q'_1 + \{w, ww_1\}, wQ_2v_2 \cup Q'_2, wQ_3v_3 \cup Q'_3, wQ_4v_4 \cup Q'_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2).

Suppose $a_5 \in H'_2$. Then we find a path Q'_2 in H_1 from v_2 to a_2 disjoint from $C_w - v_2$ and $S_3 \cap V(v_2Q_2u_2 - v_2)$. In $H_2 - Q'_2$ we find a path Q'_3 from v_3 to a_5 disjoint from $S_4 \cap V(v_3Q_3u_3 - v_3)$ and $(C_w - v_3) \cup u_3Q_3a_3$. In $H_3 - Q'_3$ we find a path Q'_4 from v_4 to a_3 disjoint from $S_1 \cap V(v_4Q_4u_4 - v_4)$ and $(C_w - v_4) \cup u_4Q_4a_4$. In $H_4 - Q'_4$, we find a path Q'_1 from w_1 to a_4 disjoint from $C_w - w_1$ and Q_1. Then $\alpha(Q'_1 + \{w, ww_1\}, wQ_2v_2 \cup Q'_2, wQ_3v_3 \cup Q'_3, wQ_4v_4 \cup Q'_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2).

Now assume $a_5 \in H'_3$. Then there is a path Q'_2 in H_1 from v_2 to a_2 disjoint from $C_w - v_2$ and $S_3 \cap V(v_2Q_2u_2 - v_2)$. In $H_2 - Q'_2$ there is a path Q'_3 from v_3 to a_3 disjoint from $C_w - v_3$ and $S_4 \cap V(v_3Q_3u_3 - v_3)$. In $H_3 - Q'_3$ there is a path Q'_4 from v_4 to a_5 disjoint from $S_1 \cap V(v_4Q_4u_4 - v_4)$ and $(C_w - v_4) \cup u_4Q_4a_4$. In $H_4 - Q'_4$, we find a path Q'_1 from w_1 to a_4 disjoint from $C_w - w_1$ and Q_1. Then $\alpha(Q'_1 + \{w, ww_1\}, wQ_2v_2 \cup Q'_2, wQ_3v_3 \cup Q'_3, wQ_4v_4 \cup Q'_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2).

Finally, assume $a_5 \in H'_4$. Then we find a path Q'_2 in H_1 from v_2 to a_2 disjoint from $C_w - v_2$ and $S_3 \cap V(v_2Q_2u_2 - v_2)$. In $H_2 - Q'_2$ we find a path Q'_3 from v_3 to a_3 disjoint from $C_w - v_3$ and $S_4 \cap V(v_3Q_3u_3 - v_3)$. In $H_3 - Q'_3$ we find a path Q'_4 from v_4 to a_4 disjoint from $C_w - v_4$ and $S_1 \cap V(v_4Q_4u_4 - v_4)$. In $H_4 - Q'_4$, we find a path Q'_1 from w_1 to a_5 disjoint from $C_w - w_1$ and Q_1. Again, $\alpha(Q'_1 + \{w, ww_1\}, wQ_2v_2 \cup Q'_2, wQ_3v_3 \cup Q'_3, wQ_4v_4 \cup Q'_4) < \alpha(Q_1, Q_2, Q_3, Q_4)$, contradicting (2) and completing the proof of (7).

Thus there exists $w' \in N(w) \cap V(v_1C_wu - u)$; for, otherwise, it follows from (7) that $\{u, u_2, u_3, u_4, v\}$ is a cut in G separating A from w, contradicting (1). We choose such w' that v_1Cw' is minimal. We now apply the arguments (3) – (7), using $ww' \cup v_1Cw' \cup v_1Q_1a_1$ (instead of Q_1), Q_2, Q_3, Q_4, and using counter clockwise order instead of clockwise order. As a consequence and by planarity, there exist $u' \in V(v_1Cw')$, $u'_2 \in V(u_2Q_2a_2)$, $u'_3 \in V(u_3Q_3a_3)$, $u'_4 \in V(u_4Q_4a_4)$, and $v' \in V(vCv_1)$ such that u' and u'_2 are cofacial, u'_2 and u'_3 are cofacial, u'_3 and u'_4 are cofacial, and u'_4 and v' are cofacial. Moreover, if H''_i denote the $\{u'_i, u'_{i+1}\}$-bridge of H_i containing $v_iC_wv_{i+1}$, $1 \le i \le 4$, with $u'_1 = u'$, $u'_5 = v'$ and $v_5 = v$, then $a_5 \notin H''_i$. However, $\{u', u'_2, u'_3, u'_4, v'\}$ is a cut in G separating A from w, contradicting (1). ∎

Theorem 4.3 *Let G be a graph drawn in a closed disc in the plane with no edge crossings, let a_1, a_2, a_3, a_4, a_5 be distinct vertices of G on the boundary of the disc in clockwise order, and let $A = \{a_1, a_2, a_3, a_4, a_5\}$. Suppose G is $(5, A)$-connected and $|V(G)| \geq 7$. Then there exist $w \in V(G) - A$, a cycle C_w in $(G - A) - w$, and four paths $P_1, \ldots, P_4$ from w to A such that*

(*i*) *$V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C_w)| = 1$ for $1 \leq i \leq 4$, and*

(*ii*) *there exist $1 \leq i \neq j \leq 4$ such that a_1 is an end of P_i and a_5 is an end of P_j.*

Proof. Assume the assertion is false, and let G be a counter example with $|V(G)|$ minimal. Then

(1) G has no 5-separation (G_1, G_2) such that $A \subseteq G_1$ and $|V(G)| > |V(G_2)| \geq 7$.

For, suppose such a separation (G_1, G_2) does exist. By Menger's theorem, there are five disjoint paths R_1, R_2, R_3, R_4, R_5 in G_1 from $V(G_1 \cap G_2)$ to A. By choosing notation appropriately, we may assume $a_i \in R_i$ for $i = 1, \ldots, 5$. Let b_i be the other end of R_i. Note that G_2 is $(5, \{b_1, b_2, b_3, b_4, b_5\})$-connected. Then by the choice of G and by appropriate notation change, there exist $w \in V(G_2) - \{b_1, b_2, b_3, b_4, b_5\}$, a cycle C_w in $G_2 - \{w, b_1, b_2, b_3, b_4, b_5\}$, and four paths Q_1, Q_2, Q_3, Q_4 in G_2 from w to $\{b_1, b_2, b_3, b_4, b_5\}$, such that $V(Q_i \cap Q_j) = \{w\}$ for $1 \leq i \neq j \leq 4$, $|V(Q_i \cap C_w)| = 1$ for $1 \leq i \leq 4$, and $b_1 \in P_i$ and $b_5 \in P_j$ for some $1 \leq i \neq j \leq 4$. Now $P_i := Q_i \cup R_i$, $i = 1, \ldots, 4$, are the desired paths.

By Lemma 4.1, $G - A$ is 2-connected. So $|V(G) - A| \geq 3$. Hence by (1), each a_i has at least two neighbors in $G - A$, and so G is 2-connected. Let C, C' denote the outer cycles of $G, G - A$, respectively. By Lemma 4.1 again, $(G - A) - C'$ is nonempty, and for each $w \in V(G - A) - V(C')$, the vertices of G that are cofacial with w induce a cycle C_w, and $C_w \subseteq G - A$. Thus, we may choose w so that *whenever possible* the following hold:

(2) if both a_1 and a_5 have exactly two neighbors on C' and a_1 and a_5 share a common neighbor x with $d(x) = 5$, then $wx \notin E(G)$, and

(3) w and a_1 have a common neighbor, or w and a_5 have a common neighbor.

By Lemma 4.2, there exist paths P_1, P_2, P_3, P_4 from w to A such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C_w)| = 1$ for $1 \leq i \leq 4$. Let $V(P_i \cap C_w) = \{w_i\}$ for $i = 1, 2, 3, 4$. If for some $1 \leq i \leq 4$, $|V(P_i) \cap A| = 2$ then we may replace it with its subpath between w_i and the vertex that is in $A \cap V(P_i)$ but is not an end of P_i. So we may assume that $A \not\subseteq \bigcup_{i=1}^{4} P_i$.

If $a_1 \in P_i$ and $a_5 \in P_j$ for some $i \neq j$, then the assertion of the theorem holds. So we may assume by symmetry (between a_1 and a_5) that $a_1 \notin P_i$ for $1 \leq i \leq 4$. By changing notation if necessary we may assume $a_2 \in P_2, a_3 \in P_3$, $a_4 \in P_4$ and $a_5 \in P_1$. See Figure 3 for an illustration.

Note that P_1, P_2, P_3, P_4 divide the disc into four closed regions. Let H_i (for each $1 \leq i \leq 4$) denote the maximal subgraph of $G - w$ contained in the closed region which has P_i and P_{i+1} in its boundary, where $P_5 = P_1$. Then $a_1 \in H_1$ (by planarity). We may further assume that the paths P_1, P_2, P_3, P_4 are chosen so that

(4) H_1 is maximal.

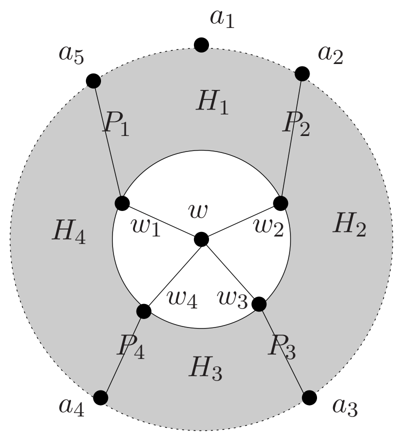

Figure 3: The regions divided by P_1, P_2, P_3, P_4.

We may assume that there does not exist a path R in H_1 from a_1 to some $a \in V(P_2)$ such that $(R-a) \cap C_w = \emptyset$, $R \cap P_1 = \emptyset$, and $R \cap P_2 = \{a\}$; for, otherwise, $P_1, wP_2a \cup R, P_3, P_4$ give the desired paths. So let v denote the vertex on $w_1C_ww_2 - w_2$ such that there is a path P in G from a_1 to v disjoint from $P_1 \cup (C_w - v)$, and subject to this, vC_ww_2 is minimal. Then $v \in C$. Also, we may assume $wv \notin E(G)$; or else $P_1, P + \{w, wv\}, P_3, P_4$ give the desired paths. We claim that

(5) $a_4Ca_5 \cap (w_4C_ww_1 - w_1) = \emptyset$.

For, otherwise, let $b \in V(a_4Ca_5) \cap V(w_4C_ww_1 - w_1)$. Then $\{b, w, v\}$ is a cut in G separating $\{w_1, a_1, a_5\}$ from $\{a_2, a_3, a_4\}$. So by (1), b and w_1 are the only neighbors of a_5 on C, v and w_1 are the only neighbors of a_1 on C, and $N(w_1) = \{a_1, a_5, b, v, w\}$. Let $w' \in N(v) \cap V(C_w - w_1)$. Since $d(v) \geq 5$ and $wv \notin E(G)$, $w' \notin C$. It is easy to see that w' contradicts the choice of w in (2) (but satisfies (3)). This completes the proof of (5).

Case 1. Suppose there exists a path Q from a_1 to $a \in V(P_1)$ such that $(Q-a) \cap C_w = \emptyset$, $Q \cap P_2 = \emptyset$, and $Q \cap P_1 = \{a\}$.

Choose Q so that aP_1w_1 is minimal. If $a_4Ca_5 \cap w_1P_1a = \emptyset$, then by (5), $P_4 \cup a_4Ca_5$ contains a path P_4' from a_5 to w such that $P_4' \cap C_w = \{w_4\}$, and so $Q \cup aP_1w, P_2, P_3, P_4'$ give the desired paths.

Hence, we may assume $a_4Ca_5 \cap w_1P_1a \neq \emptyset$. Then by (4), $aP_1a_5 = aCa_5$. By (1), $\{a, v\}$ cannot separate $\{a_5, a_1\}$ from $C_w \cup \{a_2, a_3, a_4\}$. Hence by the minimality of aP_1w_1, there is a path R in $H_1 - aCa_5$ from a_1 to some $u \in V(w_1C_wv) - \{w_1, v\}$. We choose such u that w_1C_wu is minimal. We may assume $wu \notin E(G)$; or else $R + \{w, wu\}, P_2, P_3, P_4$ give the desired paths.

Suppose w has a neighbor, say w', in $uC_wv - \{u, v\}$. Note that $\{a, u, v, w\}$ is a cut of G; and let H' denote the $\{a, u, v, w\}$-bridge of G containing a_1 and a_5. Then since G is $(5, A)$-connected and by planarity (and also because of P and R), H' contains a path R' from w' to a_1 disjoint from $(C_w - w') \cup aCa_5$. Now the assertion of the theorem holds with C_w and the paths $P_1, R' + \{w, ww'\}, P_2, P_3$.

Therefore we may assume that such w' does not exist. Then $\{a, u, v\}$ is a cut in G separating $\{a_1, a_5\}$ from $\{a_2, a_3, a_4, w\}$. So by (1), there is a vertex x such that x and a are the only neighbors of a_5 on C, x and v are the only neighbors of a_1 on C, and $N(x) = \{a_1, a_5, a, u, v\}$. Since $d(u) \geq 5$ and $wu \notin E(G)$, we see that $a \neq w_1$. So w has no common neighbor with any of a_1 and a_5. Let w' be the vertex in $N(u) \cap V(w_1 C_w u)$. Then $w'x \notin E(G)$, and w' has a common neighbor with a_1. So w' contradicts the choice of w in (3).

Case 2. There exist $u, v \in V(w_1 C_w w_2) - \{w_1, w_2\}$ such that all paths from a_1 to C_w must intersect uC_2v.

We choose such u and v so that uC_2v is minimal. Then $\{u, v, w\}$ is a 3-cut in G separating a_1 from other vertices. Since G is $(5, A)$-connected, the component of $G - \{u, v, w\}$ containing a_1 has exactly one vertex. So by planarity, u and v are the only neighbors of a_1 in G, and uv is an edge of C_w. Thus $u, v \in C$.

If $wu \in E(G)$ then wua_1, P_2, P_3, P_4 give the desired paths, and if $wv \in E(G)$ then P_1, wva_1, P_3, P_4 give the desired paths. So we may assume $wu, wv \notin E(G)$. In particular, $u \neq w_1$ and $v \neq w_2$. Without loss of generality, we may assume w_1, u, v, w_2 occur on C_w in this clockwise order.

Note that w and a_5 have no common neighbor. For, otherwise, let $b \in N(w) \cap N(a_5)$. Then since $d(u) \geq 5$ and $wu \notin E(G)$, $\{b, u, w, a_5\}$ is a cut in G, contradicting the assumption that G is $(5, A)$-connected.

Let $v_1 \in V(C_w) - \{u, v\}$ such that $v_1 v \in E(G)$. Since G is $(5, A)$-connected and $wv \notin E(G)$, $v_1 \notin C$. By Lemma 4.1, the vertices of G which are cofacial with v_1 induce a cycle in $G - A$. Note that v_1 has no common neighbor with a_5 (i.e. satisfying (2)); however, v_1 and a_1 have v as a common neighbor. So v_1 contradicts the choice of w in (3). ∎

Proof of Theorem 1.2. Let (G_1, G_2) be a 5-separation in G such that $V(G_1 \cap G_2) = \{a_1, a_2, a_3, a_4, a_5\}$ and $|V(G)| > |V(G_2)| \geq 7$. Moreover, assume that G_2 may be drawn in a closed disc in the plane without edge crossings such that a_1, a_2, a_3, a_4, a_5 occur on the boundary of the disc in clockwise order. Note that $|G_1| \geq 2$ (since G is not planar). Let $A = \{a_1, a_2, a_3, a_4, a_5\}$. We may choose (G_1, G_2) such that G_2 is maximal. Then each a_i has at least two neighbors in G_1, and A is an independent set in G_1. Hence $G_1 - a_i$ is 2-connected for $1 \leq i \leq 5$.

By Theorem 4.3, there exist $w \in V(G) - A$, a cycle C_w in $(G - A) - w$, and four paths P_1, P_2, P_3, P_4 from w to A such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C_w)| = 1$ for $1 \leq i \leq 4$. Without loss of generality, we may assume that a_i is an end of P_i, $1 \leq i \leq 4$.

If $G_1 - a_5$ contains disjoint paths A_1, A_2 from a_1, a_2 to a_3, a_4, respectively, then $C_w \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup A_1 \cup A_2$ is a TK_5 in G.

So we may assume that such A_1, A_2 do not exist. By Theorem 2.2 and by the fact that G_1 is $(5, A)$-connected, $G_1 - a_5$ can be drawn in a closed disc in the plane with no edge crossings such that a_1, a_2, a_3, a_4 occur on the boundary of the disc in this cyclic order. Let C denote the outer cycle of $G_1 - a_5$. Since G is nonplanar, a_5 has at least one neighbor, say a, such that $a \notin a_4 C a_1$.

By Theorem 4.3 there exist paths Q_1, Q_2, Q_3, Q_4 from w to A such that $V(Q_i \cap Q_j) = \{w\}$ for $1 \leq i < j \leq 4$, $|V(Q_i \cap C_w)| = 1$ for $1 \leq i \leq 4$, a_4 is an end of Q_3, and a_5 is an end of Q_4. Let a_s, a_t be the ends of Q_1, Q_2 with $1 \leq s < t \leq 3$. If $(G_2 - a_5) - a_4 C a_s$ has a path R from a to a_t, then $C_w \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup (R \cup a_5 a) \cup a_4 C a_s$ is a TK_5 in G.

So we may assume that such a path R does not exist. Then $s = 2$ and $a \in a_1 C a_s$. By Theorem 4.3 there exist paths R_1, R_2, R_3, R_4 from w to A such that $V(R_i \cap R_j) = \{w\}$ for

$1 \leq i < j \leq 4$, $|V(R_i \cap C_w)| = 1$ for $1 \leq i \leq 4$, a_1 is an end of R_1, and a_5 is an end of R_4. Let a_s, a_t be the ends of R_2, R_3 with $2 \leq s < t \leq 4$. Now $C_w \cup R_1 \cup R_2 \cup R_3 \cup R_4 \cup a_t C a_1 \cup (a_5 a \cup a C a_s)$ is a TK_5 in G. ∎

References

[1] P. Catlin, Hajós' graph-coloring conjecture: variations and counterexamples, *J. Combin. Theory, Ser. B* **26** (1979) 268–274.

[2] K. Chakravarti and N. Robertson, Covering three edges with a bond in a nonseparable graph. *Annals of Discrete Math.* (Deza and Rosenberg eds) (1979) 247.

[3] G. Chen, R. Gould, and X. Yu, Graph connectivity after path removal, *Combinatorica* **23** (2003) 185–203.

[4] S. Curran and X. Yu, Non-separating cycles in 4-connected graphs, *SIAM J. Discrete Math.* **16** (2003) 616–629.

[5] R. Diestel, *Graph Theory (3rd edition)*, Graduate Text in Mathematics 173, Springer, 2006.

[6] G. A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc., Ser. B* **27** (1952) 85-92.

[7] K. Kawarabayashi, Note on k-contractible edges in k-connected graphs, *Australas. J. Combin.* **24** (2001) 165–168.

[8] K. Kawarabayashi, Contractible edges and triangles in k-connected graphs, *J. Combin. Theory Ser. B* **85** (2002) 207–221.

[9] K. Kawarabayashi, O. Lee, and X. Yu, Non-separating paths in 4-connected graphs, *Annals of Combinatorics* **9** (2005) 47–56.

[10] A. K. Kelmans, Every minimal counterexample to the Dirac conjecture is 5-connected, *Lectures to the Moscow Seminar on Discrete Mathematics* (1979).

[11] A. E. Kézdy and P. J. McGuiness, Do $3n - 5$ edges suffice for a subdivision of K_5? *J. Graph Theory* **15** (1991) 389-406.

[12] M. Kriesell, Induced paths in 5-connected graphs, *J. Graph Theory* **36** (2001) 52–58.

[13] L. Lovász, Problems, in *Recent Advances in Graph Theory*, M. Fiedler, ed., Academia, Prague, 1975.

[14] W. Mader, $3n - 5$ Edges do force a subdivision of K_5, *Combinatorica* **18** (1998) 569-595.

[15] N. Robertson, P. D. Seymour, and R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica* **13** (1993) 279–361.

[16] P. D. Seymour, Private Communication with X. Yu.

[17] P. D. Seymour, Disjoint paths in graphs. *Discrete Math.* **29** (1980) 293–309.

[18] Y. Shiloach, A polynomial solution to the undirected two paths problem, *J. Assoc. Comp. Mach.* **27** (1980) 445–456.

[19] C. Thomassen, 2-Linked graphs. *Europ. J. Combin.* **1** (1980) 371–378

[20] W. T. Tutte, How to draw a graph, *Proc. London Math. Soc.* (3) **13** (1963) 743–767.

[21] M. E. Watkins and D. M. Mesner, Cycles and connectivity in graphs, *Canadian J. Math.* **19** (1967) 1319-1328.

[22] X. Yu, Subdivisions in planar graphs. *J. Combin. Theory Ser. B* **72** (1998) 10–52.

[23] X. Yu and F. Zickfeld, Reducing Hajos' coloring conjecture to 4-connected graphs, *J. Combin. Theory Ser. B* **96** (2006) 482–492.